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On the generation of electrovac solutions from vacuum solutions in general relativity

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Abstract. The prescription given by Gautreau and Hoffman to generate axially symmetric electrovac solutions from the Weyl vacuum solutions in general relativity, is shown to be completely equivalent to the procedure given earlier by Harrison yielding, thus, only a known class of solutions of the Einstein–Maxwell equations.

It is well known (Synge 1960) that every static line element that depends on at most two variables (r, z) can be reduced to the canonical isothermal form

$$ds^2 = -dt^2 \exp(2U)+(dr^2+dz^2)\exp(2V-2U)+r^2\, d\phi^2 \exp(-2U), \qquad (1)$$

where U and V are functions of r and z only. In discussing the solutions of the field equations of general relativity which correspond to the line element given above, it is convenient to introduce two functions $\lambda \equiv \lambda(r, z)$ and $\nu \equiv \nu(r, z)$ defined by the system of equations

$$\Delta\lambda+\lambda_1/r = 0 \qquad (2)$$

$$\nu_1 = r(\lambda_1^2-\lambda_2^2), \qquad \nu_2 = 2r\lambda_1\lambda_2 \qquad (3)$$

$$\Delta\nu+\lambda_1^2+\lambda_2^2 = 0 \qquad (4)$$

where $\Delta\lambda \equiv \lambda_{11}+\lambda_{22}, \Delta\nu \equiv \nu_{11}+\nu_{22}$, the subscripts on the right of λ and ν indicate partial derivatives with respect to $x^1 = r$ and $x^2 = z$ and the superscripts denote, as usual, the powers. Apparently there are three equations in (2), (3) and (4), but when (2) is satisfied, (3) are integrable and (4) is implied by the other two equations (2) and (3).

In this note, we consider those classes of electrovac solutions which can be generated from the solutions of the empty space field equations $R_{\mu\nu} = 0$. As is well known, the empty space field equations for the line element (1) are precisely the equations (2), (3) and (4) with λ replaced by U and ν replaced by V. Hence, it follows that with $U = \lambda$ and $V = \nu$, the metric (1) is a solution of the vacuum field equations provided, in addition, the conditions of elementary flatness are also satisfied (Synge 1960). To investigate non-null electrovac fields that correspond to (1), we can either use the coupled Einstein–Maxwell equations in their conventional form or the equivalent Rainich–Misner–Wheeler (RMW) equations (Misner and Wheeler 1957). Here, we adopt the RMW formalism as it does not require making special assumptions about the nature of the electromagnetic field and keeps geometry to the fore. Using (1) and the corresponding components of the Ricci tensor which are quoted in Synge (1960), it can be shown that the Rainich algebraic conditions reduce to (Gopala Rao and Srinivasa Rao 1973)

$$\Delta V = \Delta U + U_1/r - U_1'^2 - U_2^2 \tag{5}$$

$$(\Delta U + U_1/r)^2 - (U_1^2 - U_2^2 - V_1/r)^2 = (2U_1U_2 - V_2/r)^2. \tag{6}$$

A direct calculation shows that the complexion of the electromagnetic field is an arbitrary constant so that the differential RMW equations are automatically satisfied. Hence, any set of solutions (U, V) of the equations (5) and (6) which yields a positive value for R_0^0 (this is necessary to assure the positiveness of the electromagnetic field energy), gives an electrovac solution for the metric (1). Once U and V are so determined, one can immediately calculate the electromagnetic fields and also obtain some information about the nature of the sources of the fields represented by (1) (Gopala Rao and Srinivasa Rao 1973).

A number of special solutions of the equations (5) and (6) have been discussed in literature. These solutions include, in particular, those which can be generated from the vacuum functions λ and ν which were introduced earlier. For example, the results of Harrison (1965) when applied to the metric (1), imply that to every solution (λ, ν) of the equations (2), (3) and (4), there corresponds a solution (U, V) of the RMW equations (5) and (6) given by

$$U \equiv U(\lambda) = \lambda + \ln[1 + r^2 \exp(-2\lambda)] - \ln(2A) \tag{7}$$

$$V \equiv V(\lambda, \nu) = \nu + 2\ln[1 + r^2 \exp(-2\lambda)] - 2\ln(2A) \tag{8}$$

where A is an arbitrary constant. Similarly Gautreau and Hoffman (1970) have shown that

$$U = \ln(Br\cosh\lambda) \tag{9}$$

$$V = \ln(r\cosh^2\lambda) + \nu \tag{10}$$

with B an arbitrary constant, satisfy (5) and (6). We will now demonstrate that these two prescriptions for generating electrovac solutions are one and the same.

Let λ and μ be any two solutions of the Laplace equation (2). Then, obviously $\tilde{\lambda} = \lambda + \mu$ is also a solution of (2) and it follows from (3) and (4) that the function $\tilde{\nu}$ which corresponds to $\tilde{\lambda}$ is given by

$$\tilde{\nu}_1 = \nu_1 + r(\mu_1^2 - \mu_2^2) + 2r(\lambda_1\mu_1 - \lambda_2\mu_2) \tag{11}$$

$$\tilde{\nu}_2 = \nu_2 + 2r\mu_1\mu_2 + 2r(\lambda_1\mu_2 + \lambda_2\mu_1) \tag{12}$$

$$\Delta\tilde{\nu} + (\lambda_1 + \mu_1)^2 + (\lambda_2 + \mu_2)^2 = 0. \tag{13}$$

Choosing $\mu = \ln r$, we have $\exp(\tilde{\lambda}) = r\exp(\lambda)$, and on integrating (11) and (12), we get

$$\tilde{\nu} = \nu + 2\lambda + \ln(fr), \qquad f = \text{constant}. \tag{14}$$

Clearly, the equation (13) is automatically satisfied. Now, using the Harrison prescription (7) and (8), we get

$$U(\tilde{\lambda}) = \ln(Br\cosh\lambda)$$

$$V(\tilde{\lambda}, \tilde{\nu}) = \ln(r\cosh^2\lambda) + \nu + \ln(B^2 f)$$

where we have written $A = 1/B$. Choosing $B^2 = 1/f$, we see that this is precisely the prescription of Gautreau and Hoffman given in (9) and (10).

Thus, it is now evident that we may generate many such formulae by using suitable superposed solutions of the Laplace equation (2). As an example, consider

$$U = -\ln[a\exp(\lambda)+b\exp(-\lambda)] \tag{15}$$

$$V = \nu \tag{16}$$

where a and b are constants. These are clearly solutions of (5) and (6) and form the content of the Weyl prescription (see Gautreau *et al* 1972) to generate electrovac solutions from the vacuum functions λ and ν. With $\lambda \to \lambda+\ln r$, we obtain from (14), (15) and (16), the 'new' prescription

$$U = -\ln\left(ar\exp(\lambda)+\frac{b}{r}\exp(-\lambda)\right)$$

$$V = \ln[fr\exp(2\lambda+\nu)]$$

to generate electrovac solutions from vacuum solutions.

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References

Gautreau R and Hoffman R B 1970 *Phys. Rev.* D **2** 271–2
Gautreau R, Hoffman R B and Armeni A Jr 1972 *Nuovo Cim.* B **7** 71–98
Gopala Rao A V and Srinivasa Rao K N 1973 *Proc. Ind. Natn. Sci. Acad.* in the press
Harrison B K 1965 *Phys. Rev.* **138** B488–94
Misner C W and Wheeler J A 1957 *Ann. Phys., NY* **2** 525–603
Synge J L 1960 *Relativity: The General Theory* (Amsterdam: North-Holland) pp 311–2